Bessel Functions, Recursion and a Nonlinear Field Equation

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We show that particular solutions of certain nonlinear field equations can be constructed using Bäcklund transformations, recursion and Bessel functions.

Key words: Bessel Function; Thirring Model; Bäcklund Transformation.

It is well-known that the modified Bessel differential equation [1]

$$\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} - \left(1 + \frac{n^2}{r^2}\right) w = 0 \qquad (1)$$

admits as solution the modified Bessel function

$$I_n(r) = \left(\frac{1}{2} r\right)^n \sum_{s=0}^{\infty} \frac{\left(\frac{1}{4} r^2\right)^s}{n!\,\Gamma(n+s+1)}. \qquad (2)$$

The modified differential equation (1) also admits the Macdonald's function as a solution. This solution is not considered here. The modified Bessel function I_n satisfies the recurrence relation [1]

$$I_{n+1}(r) = -\frac{n}{r} I_n(r) + \frac{dI_n(r)}{dr}. \qquad (3)$$

Thus, if $n = 0$ we have $I_1 = dI_0/dr$ and for $n = 1$ we have

$$I_2 = -\frac{1}{r} I_1 + \frac{dI_1}{dr} = -\frac{1}{r}\frac{dI_0}{dr} + \frac{d^2 I_0}{dr^2}. \qquad (4)$$

For $n = 0$ the modified Bessel differential equation (1) takes the form

$$\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr} - w = 0. \qquad (5)$$

The Riccati differential equation

$$\frac{d\Phi_1}{dr} + \frac{1}{r}\Phi_1 = 1 - \Phi_1^2 \qquad (6)$$

is a nonlinear differential equation of first order and can be linearized using the transformation [2]

$$\Phi_1 = \frac{1}{w}\frac{dw}{dr}, \qquad (7)$$

where w satisfies the modified Bessel differential equation with $n = 0$. Thus

$$\Phi_1(r) = \frac{I_1(r)}{I_0(r)} \qquad (8)$$

is a solution of the Riccati differential equation. We can now use the relations to construct solutions of certain nonlinear field equations which admit Bäcklund transformations [3].

As an example we consider the nonlinear complex field equation [4]

$$\Delta\psi + \frac{(\nabla\psi)^2\,\overline{\psi}}{1 - |\psi|^2} + \psi(1 - |\psi|^2) = 0, \qquad (9)$$

where $\Delta := \partial^2/\partial x^2 + \partial^2/\partial y^2$. Equation (9) admits the Bäcklund transformation

$$i\frac{\partial u}{\partial \bar{z}} + v - |u|^2 v = 0, \qquad (10a)$$

$$i\frac{\partial v}{\partial z} + u - |v|^2 u = 0, \qquad (10b)$$

when $z = (x - iy)/2$, $\bar{z} = (x - iy)/2$, and u and v satisfy (9). We have $\partial/\partial z = \partial/\partial x - i\partial/\partial y$ and $\partial/\partial\bar{z} = \partial/\partial x + i\partial/\partial y$. System (10) is the Euclidian version of the massive Thirring model [4]. The system of partial differential equations (10) can be derived from the Lagrangian [3]

$$\mathcal{L} = i\bar{u}\frac{\partial v}{\partial z} + i\bar{v}\frac{\partial u}{\partial \bar{z}} + |u|^2 + |v|^2 - |uv|^2 - 1 + c.c. \qquad (11)$$

Inserting the ansatz

$$\psi(x, y) = \Phi_n(r)\exp(in\theta) \qquad (12)$$

into the nonlinear equation (9), where r, θ are polar coordinates and Φ_n is real valued, yields

$$\frac{d^2\Phi_n}{dr^2} + \frac{1}{r}\frac{d\Phi_n}{dr} + \frac{\Phi_n}{1 - \Phi_n^2}\left[\left(\frac{d\Phi_n}{dr}\right)^2 - \frac{n^2}{r^2}\right] + \Phi_n(1 - \Phi_n^2) = 0. \qquad (13)$$

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Let $u = -i\Phi_{n-1}\exp(i(n-1)\,\theta)$ and $v = \Phi_n \exp(in\theta)$. Then we obtain the equivalent system

$$\frac{\mathrm{d}\Phi_{n-1}}{\mathrm{d}r} - \frac{n-1}{r}\Phi_{n-1} + \Phi_n(1-\Phi_{n-1}^2) = 0, \quad (14a)$$

$$\frac{\mathrm{d}\Phi_n}{\mathrm{d}r} + \frac{n}{r}\Phi_n - \Phi_{n-1}(1-\Phi_n^2) = 0, \quad (14b)$$

where Φ_n and Φ_{n-1} satisfy (12) with n and $n' = n-1$, respectively. For $n = 1$, (14a) is solved by $\Phi_0 = 1$. For this case (14b) takes the form of the Ricatti equation (6), which admits the solution (8). Since

$$\Phi_{n+1} = \frac{-1}{1-\Phi_n^2}\left[\frac{\mathrm{d}\Phi_n}{\mathrm{d}r} - \frac{n}{r}\Phi_n\right] = \Phi_{n-1} - \frac{2}{1-\Phi_n^2}\frac{\mathrm{d}\Phi_n}{\mathrm{d}r}, \quad n = 1, 2, \ldots, \quad (15)$$

we obtain

$$\Phi_2 = -\frac{I_0 I_2 - I_1^2}{I_0^2 - I_1^2},$$
$$\Phi_3 = \frac{(I_3 - I_1)(I_0^2 - I_1^2) + I_1(I_0 - I_2)^2}{(I_0 - I_2)(I_0 I_2 - 2I_1^2 + I_0^2)}, \quad (16)$$

and so on, where we have used relation (3).

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